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# The Lens Condition for Factor Price Equalizaton

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Deardorff (1994) provides a condition that is necessary for factor price equalization across countries. That condition is a generalization of "country endowments contained in the diversification cone" from the standard 2×2×2 Heckscher-Ohlin model to the case of many goods, countries and factors. He also shows that this condition is *sufficient* in the case of two countries and conjectures that the sufficiency might hold in general. In this paper we establish the sufficiency in further cases. However, we show by a counterexample that the sufficiency does not hold in general.

**keywords**: factor price equalization, integrated world economy, lens condition

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# 1 Introduction

An integrated world economy approach to factor price equalization (FPE) in the Heckscher-Ohlin (HO) model shows the following: an international economy has an equilibrium with equalized factor prices if and only if the integrated world economy has an equilibrium whose outputs can be produced by the countries each using only their own factor endowments and using the integrated world economy techniques (FPE Condition). When are such conditions satisfied? Discussion in the literature (Deardorff (1994), Dixit and Norman (1980)) relates the FPE Condition to the condition that factor endowments of countries do not differ as much as the factor intensity variations at the integrated economy equilibrium.

In the 2 country, 2 good, 2 factor Heckscher-Ohlin model, the FPE condition is equivalent to the condition that the country endowments are in the cone of diversification defined by the factor use vectors at the integrated world equilibrium. The diversification cone forms a parallelogram when drawn from the lower-left and upper-right corners of an Edgeworth Box. Deardorff (1994) generalizes this to the case of many countries, goods, and factors. In higher dimensions, the parallelogram takes the shape of a "lens," while the country endowments determine another lens. The condition in the $2 \times 2 \times 2$ world that "the country endowment point must be in the parallelogram of diversification" is replaced by the condition that "the country lens must lie inside the goods lens" (Lens condition).

Deardorff shows that the Lens Condition is *necessary* for the FPE Condition and that it is also *sufficient* in the case of two countries. He then conjectures that sufficiency might hold in general. We show that the Lens

Condition is also sufficient when there are at least as many factors as goods (under a full rank condition on the factor use matrix). However, we show by a counterexample that the Lens Condition is not sufficient for the FPE Condition in general.

## 2 Notation and Definitions

The framework is a multi-dimensional HO model; countries share identical constant returns to scale technologies, markets are perfectly competitive, factors are mobile within countries – but not across countries – and trade is free. In the approach of Dixit and Norman (1980) to factor price equalization, a central concept is the "Integrated World Economy" (IWE), a hypothetical economy obtained by allowing factors (as well as goods) to move freely across countries in the above model. (The standard HO model will henceforth be referred to as the "international economy," to distinguish it from the IWE.)

There are $m$ countries, $n$ goods and $f$ factors. Countries and goods are indexed by $i$ and $j$ respectively. The $f$-vector $v_j$, $j = 1, \cdots, n$ is the *row* vector of factors used in the IWE equilibrium production of good $j$. The $f$-vector $V_i$, $i = 1, \cdots, m$, is the *row* vector of factor endowment of country $i$. The $n \times f$ matrix $\mathbf{v}$ and $m \times f$ matrix $\mathbf{V}$ are then defined by

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} V_1 \\ \vdots \\ V_m \end{bmatrix}$$

We assume that these matrices do not contain zero rows or columns to exclude trivial cases. Let $u_n = (1, 1, \cdots, 1)$ be the $n$-vector of 1s. We denote by $I_m$ the set of *row* vectors in $R^m$ whose coordinate values are between zero and one. Thus, $I_m$ is the $m$-dimensional unit cube. A vertex of $I_m$ is an element of $I_m$ that has either 0 or 1 in each of its coordinates. Finally, $I_{m\times n}$ represents the set of $m$ by $n$ matrices with elements in $[0,1]$. Assuming that all factors are fully employed at the IWE equilibrium, the sum of factors used to produce the IWE output equals the sum of country endowments. That is, $u_n\mathbf{v} = u_m\mathbf{V}$.

If the factor prices are equalized in an international economy equilibrium, factors have no incentive to move across borders. Hence, those prices and production and consumption vectors from the international economy equilibrium also constitute an equilibrium of the IWE. Therefore, if the factor prices are equalized in an international economy equilibrium, the IWE has an equilibrium that can be replicated in the international economy. Conversely, if it is possible to replicate an IWE equilibrium *production* in the international economy using the techniques of the IWE equilibrium (i.e., using the same factor intensities as in the IWE equilibrium), the IWE equilibrium prices, together with these production allocations, give an international economy equilibrium with equalized factor prices[1]. Replicating IWE production suffices because replicating IWE *consumption* does not pose a difficulty due to free trade.

To replicate production of the IWE, we need to assign each country $i$ an output share $\lambda_i$ ($\lambda_i$ in $I_n$) where $\lambda_{ij}$ denotes the share of country $i$ in world production of good $j$. To make sure that production of each good in the international economy is the same as that in the IWE, country shares must satisfy $\sum_{i=1}^{m} \lambda_i = u_n$. Furthermore, if countries use the IWE techniques, these

[1]See Dixit and Norman(1980,110-122) for details.

output assignments require factor inputs $\lambda_i \mathbf{v}$ from country $i$. For this to be feasible, we need $\lambda_i \mathbf{v} \leq V_i$. Since all factors are assumed to be employed in the IWE equilibrium, $\lambda_i \mathbf{v} = V_i$. The foregoing can be compactly written as:

**Definition 1** *(FPE Condition) There exists $\boldsymbol{\lambda}$ in $I_{m \times n}$ satisfying* (1) $\boldsymbol{\lambda}\mathbf{v} = \mathbf{V}$ *and* (2) $u_m \boldsymbol{\lambda} = u_n$.

An international economy has an equilibrium with FPE if and only if the FPE Condition holds for some IWE equilibrium. If both the IWE and the international economy has a unique equilibrium, FPE obtains if and only if the FPE Condition holds.

The following definition is due to Deardorff (1994).

**Definition 2** *(lens) The lens of an $n \times f$ matrix $v$, $\mathcal{L}(\mathbf{v})$, is:*

$$\mathcal{L}(\mathbf{v}) = \{b\mathbf{v} \mid b \in I_n\}.$$

His main result is below:

**Theorem (**Deardorff (1994)**)**

$$\text{FPE Condition} \quad \Longrightarrow \quad \mathcal{L}(\mathbf{V}) \subset \mathcal{L}(\mathbf{v}).$$

The condition $\mathcal{L}(\mathbf{V}) \subset \mathcal{L}(\mathbf{v})$ will be referred to as the *Lens Condition*. It tells us that the lens constructed using the country factor endowment vectors lies inside the lens constructed using the factor use vectors. It is often convenient in the proofs to use The Lens Condition in the following form:

$$\text{Given any } a \text{ in } I_m, \text{ there is } b \text{ in } I_n \text{ satisfying } b\mathbf{v} = a\mathbf{V}.$$

Deardorff conjectures that the reverse of the above theorem might hold.

**Deardorff's Conjecture**:

$$\text{FPE Condition} \quad \Longleftarrow \quad \mathcal{L}(\mathbf{V}) \subset \mathcal{L}(\mathbf{v}).$$

## 3 Deardorff's Conjecture in Special Cases

The Deardorff conjecture is valid in several special cases. The first is given by Deardorff (1994) and is also implicit in the discussion of Dixit and Norman (1980).

**Proposition 1** *(Deardorff) The Lens Condition implies the FPE Condition when there are only two countries.*

We give further cases where the Lens Condition is sufficient for the FPE Condition. Our first result covers the subcase where the number of goods is equal to the number of factors and the factor use matrix is nonsingular.

**Proposition 2** *The Lens Condition implies the FPE Condition when there are at least as many factors as goods ($n \leq f$) and $\mathbf{v}$ has full rank.*

**Proof.** Each row $V_i$ of $\mathbf{V}$ is in $\mathcal{L}(\mathbf{V})$. Since $\mathcal{L}(\mathbf{V}) \subset \mathcal{L}(\mathbf{v})$, $V_i$ is also in $\mathcal{L}(\mathbf{v})$. By the definition of lens, there is $\lambda_i$ in $I_n$ satisfying $\lambda_i \mathbf{v} = V_i$, for each country $i$. Define a $m \times n$ matrix

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}.$$

Then $\boldsymbol{\lambda}$ is in $I_{m\times n}$ and $\boldsymbol{\lambda}\mathbf{v} = \mathbf{V}$. Since $\mathbf{v}$ has full rank, it has $n$ columns that are linearly independent. Without loss of generality, let the first $n$ columns of $\mathbf{v}$ be linearly independent. Then, we can write $\mathbf{v}$ as a partitioned matrix $\mathbf{v} = [\mathbf{v}_A|\mathbf{v}_B]$ where $\mathbf{v}_A$ is a $n \times n$ invertible matrix. Similarly, we partition $\mathbf{V}$ as $\mathbf{V} = [\mathbf{V}_A|\mathbf{V}_B]$ in a conforming manner. Trivially, since $\boldsymbol{\lambda}\mathbf{v} = \mathbf{V}$, $\boldsymbol{\lambda}\mathbf{v}_A = \mathbf{V}_A$ and since $u_n\mathbf{v} = u_m\mathbf{V}$, $u_n\mathbf{v}_A = u_m\mathbf{V}_A$. Thus, $\boldsymbol{\lambda} = \mathbf{V}_A\mathbf{v}_A^{-1}$ and $u_m\boldsymbol{\lambda} = u_m\mathbf{V}_A\mathbf{v}_A^{-1} = (u_n\mathbf{v}_A)\mathbf{v}_A^{-1} = u_n$. ■

When there are only two goods, the full rank condition of $\mathbf{v}$ in Proposition 2 can be dropped:

**Corollary 1** *The Lens Condition implies the FPE Condition when there are only two goods.*

**Proof.** First, consider the case of $n = 2 \leq f$. If the rank of $\mathbf{v}$ is 2, the conclusion follows from Proposition 2. The rank of $\mathbf{v}$ is not zero since it contains nonzero elements. Suppose the rank of $\mathbf{v}$ is 1. Since the rows $v_1, v_2$ are nonzero and nonnegative, $v_2 = \delta v_1$, for some $\delta > 0$. Under the Lens Condition, there is $b_i$ in $I_2$ such that $b_i\mathbf{v} = V_i$, for each country $i$. Thus, for each country $i$, $V_i = b_i\mathbf{v} = \gamma_i v_1$, for some $\gamma_i > 0$. Since $(1+\delta)v_1 = u_2\mathbf{v} = u_m\mathbf{V} = \sum_i \gamma_i v_1$, $1 + \delta = \sum_i \gamma_i$. If we choose $\lambda_i = (\gamma_i / \sum_i \gamma_i)u_2$, for each $i$, $\lambda_i\mathbf{v} = \gamma_i v_1 = V_i$, for each $i$ and $\sum_i \lambda_i = u_2$. Finally, if $f = 1$, the above argument applies trivially. ■

## 4 Counterexamples for Deardorff's Conjecture

First, we prove a lemma that simplifies the verification of the Lens Condition. For the Lens Condition to be satisfied, it is enough that it is satisfied at all vertices of $I_m$.

**Lemma 1** *The equation* $b\mathbf{v} = a\mathbf{V}$ *has a solution* $b$ *in* $I_n$ *for each* $a$ *in* $I_m$ if and only if *the equation has a solution* $b$ *in* $I_n$ *for each vertex* $a$ *in* $I_m$.

**Proof.** The only if part is trivial. For the if part, suppose that for each vertex $a_i$ of $I_m$, there exists $b_i$ in $I_n$ such that $b_i\mathbf{v} = a_i\mathbf{V}$, and consider an arbitrary $a$ in $I_m$. We need to find $b$ in $I_n$ such that $b\mathbf{v} = a\mathbf{V}$. The cube $I_m$ is the set of convex combinations of its vertices. Thus, $a$ can be written as $a = \sum_i \alpha_i a_i$, where $\{a_i\}$ are the vertices of $I_m$ and $\{\alpha_i\}$ are numbers in $[0, 1]$ that sum up to one. Let $b \equiv \sum_i \alpha_i b_i$. Since $I_n$ is convex, $b = \sum_i \alpha_i b_i$ is in $I_n$, and $a\mathbf{V} = \sum_i \alpha_i a_i \mathbf{V} = \sum_i \alpha_i b_i \mathbf{v} = b\mathbf{v}$. ■

## 4.1 A Counterexample

The following example shows that the Deardorff Conjecture is not generally valid. Suppose that there are 4 goods, 3 factors and 3 countries. Suppose that the factor content of the IWE equilibrium production $\mathbf{v}$ and the endowment of countries $\mathbf{V}$ are:

$$\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{V} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We show below that 1) $u_n\mathbf{v} = u_m\mathbf{V}$, 2) $\mathcal{L}(\mathbf{V}) \subset \mathcal{L}(\mathbf{v})$ and 3) The FPE Condition is not satisfied. These establish a counterexample to the Deardorff Conjecture.

**Interpretation**

There are three factors, $K$, $L$ and $T$ (capital, labor and land) and the world factor endowment of each of them is 2 units. The first good uses only $K$, the second only $L$, and the third only $T$. The fourth good uses all the three factors $K$, $L$ and $T$. There are three countries, each of them lacking one of the three factors. As a result, none of the countries can produce good 4, although the IWE could produce it. However, we show that the lens condition is satisfied for these matrices.

**Proof of the counterexample:**

1) $u_n\mathbf{v} = (2,2,2) = u_m\mathbf{V}$.

2) When $m = 3$, the vertices of $I_3$ are: $0_3 = (0,0,0)$, $a_1 = (1,0,0)$, $a_2 = (0,1,0)$, $a_3 = (0,0,1)$, $a_{12} = (1,1,0)$, $a_{13} = (1,0,1)$, $a_{23} = (0,1,1)$ and $u_3$. We have $u_4\mathbf{v} = u_3\mathbf{V}$ and $0_4\mathbf{v} = 0_3\mathbf{V}$, where $0_4$ is the 4-dimensional zero vector. Thus, to check the Lens Condition, we only need to look for $b_1$, $b_2$, $b_3$, $b_{12}$, $b_{13}$, $b_{23}$ in $I_4$ solving:

$$\begin{aligned} b_1\mathbf{v} &= a_1\mathbf{V}, \quad b_{12}\mathbf{v} = a_{12}\mathbf{V} \\ b_2\mathbf{v} &= a_2\mathbf{V}, \quad b_{13}\mathbf{v} = a_{13}\mathbf{V} \\ b_3\mathbf{v} &= a_3\mathbf{V}, \quad b_{23}\mathbf{v} = a_{23}\mathbf{V} \end{aligned} \tag{1}$$

The following values satisfy the equations in (1) and thus the Lens Condition is satisfied by Lemma 1: $\{b_1 = (0,1,1,0)\,,\ b_2 = (1,0,1,0)\,,\ b_3 = (1,1,0,0)\,, b_{12} = (0,0,1,1)\,, b_{13} = (0,1,0,1)\,, b_{23} = (1,0,0,1)\}$. We can check that these are solutions by verifying:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

3) Since good 4 cannot be produced by any of the countries, IWE production cannot be duplicated by this endowment distribution. Formally, suppose that $\boldsymbol{\lambda}$ in $I_{3\times4}$ satisfies the equation $\boldsymbol{\lambda}\mathbf{v} = \mathbf{V}$. Consider $\lambda_{i4}$, the $i$th element of the fourth column. Since $\boldsymbol{\lambda}$ is in $I_{3\times4}$, $\lambda_{i4} \geq 0$. If $\lambda_{i4} > 0$, the $i$th row of $\boldsymbol{\lambda}\mathbf{v}$ is strictly positive while each row of $\mathbf{V}$ has a zero element. Hence, $\lambda_{i4} = 0$ for all $i \in \{1, 2, 3\}$. Then, $\lambda_{14} + \lambda_{24} + \lambda_{34} = 0$. This contradicts the requirement that country shares add up to 1. ■

We include a figure to help visualize the lenses $\mathcal{L}(\mathbf{v})$ and $\mathcal{L}(\mathbf{V})$ in this counterexample. Since there are three factors, both $\mathbf{v}$ and $\mathbf{V}$ have three columns. The lenses $\mathcal{L}(\mathbf{v}) = \{b\mathbf{v} \mid b \in I_n\}$ and $\mathcal{L}(\mathbf{V}) = \{a\mathbf{V} \mid a \in I_m\}$ are subsets of $R^3$. Each dimension of the three dimensional cubes in the figure corresponds to a different factor. The point $(2, 2, 2)$ at the far-upper-right corner of each box corresponds to the aggregate world endowment.

## 4.2 Another Counterexample:

We give another counterexample for comparison. Its verification is similar to the previous one.

$$\mathbf{v} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

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Figure 1: Lenses for the counterexample in section 4.1