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The Sufficiency of the "Lens Condition" for Factor Price Equalization in the Case of Two Factors

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Abstract:

Factor price equalization (FPE) is a central theme in trade theory, for which Dixit and Norman (1980) establish the necessary and sufficient condition (the FPE condition). Deardorff (1994) provides a more intuitive condition (the lens condition) and establishes its necessity in general, as well as its sufficiency for the case of 2 countries. In this paper, I prove that the lens condition is sufficient for FPE in the case of 2 factors. This theorem has implications for empirical work.

Hanson, who gave me very helpful comments.

1. Introduction

The concept of factor price equalization (FPE) is central to the theory of international trade. Dixit and Norman (1980) establish that with perfect competition and constant returns to scale, the necessary and sufficient condition for FPE is that the outputs of the "integrated world economy" (IWE) can be produced using the technology of the IWE, with each factor of each country fully employed. [1]To help visualize this somewhat abstract condition, a factor endowment box (FEB)[2] is presented both for the case of 2 factors, 2 countries and 2 goods and that of 2 factors, 2 countries and 3 goods. In the FEB, a polygon (in the case of 2 goods, a parallelogram) can be drawn, with each side being the factor usage vector of one sector in the IWE. When there are only 2 countries, the distribution of factor endowments can be represented by a point in the FEB. It is then straightforward that FPE is achieved if and only if the point lies within the polygon. A natural question to ask is, if the numbers of factors, goods and countries are arbitrary, can we still visualize the FPE condition in a similar fashion?

Deardorff (1994) explores this idea and formalizes the visualization as the "lens condition"; i.e. the lens spanned by the vectors of the countries' factor endowments lies inside the lens spanned by the factor usage vectors of all sectors in the IWE. In addition to being easier to visualize than the Dixit-Norman condition, the lens condition also formalizes the intuition that, to achieve FPE, the variation across countries in relative endowments must be less, in some sense, than the variation across industries in factor intensities. Deardorff proceeds to show that the lens condition is necessary for FPE in general, and in the case of 2 countries it is also sufficient. He also conjectures that the lens condition is sufficient for an arbitrary number of factors, goods and countries. (I refer this conjecture as the " Deardorff conjecture".)

In this paper, I show that the conjecture holds in the case of 2 factors for an arbitrary number of goods and countries. The case of 2 factors deserves attention because the factor endowment box has probably become the most widely used tool to analyze factor price equalization, and in almost all cases, such a box is drawn in 2 dimensions. It is therefore important to notice, and comforting to know, that such a box drawn in 2 dimensions is indeed appropriate for an arbitrary number of goods and countries. What's more, the theorem proved in this paper could serve as the theoretical foundation for meaningful empirical work. For instance, some authors have noticed that the Heckscher-Ohlin-Vanek (HOV) theorem performs poorly even among developed countries.[3] Could this be because even developed countries have very different factor endowments so that FPE fails to hold among them? This is an empirical question, and one good way of answering it is to test the lens condition. However, if the lens condition is necessary but insufficient, the answer is incomplete. The rejection of the lens condition implies that the answer is yes, but the acceptance of the lens condition does not imply that the answer is no. Only when the lens condition is both necessary and sufficient can we obtain an unambiguous answer in both cases.

Debaere and Demiroglu (1998) test the lens condition empirically for 2 factors (capital and labor) and for different groups of countries.[4] They accept the lens condition for a group of developed countries, and reject the condition for a group consisting of both developed and developing countries. Their work suggests that among developed countries, factor endowments are similar enough for FPE to hold.[5] Therefore it seems appropriate to conclude that the poor

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[1] Blackorby, Schworm and Venables (1993) provide alternative necessary and sufficient conditions for factor price equalization.

[2] Figure 4.4 and 4.6 in Dixit and Norman (1980).

[3] See, for example, Brecher and Choudhri (1988), and Gabaix (1997).

[4] They cited Demiroglu, Ufuk (1997) "The lens Condition: The Two Factor Case" (mimeo, University of Michigan) and Qi, Ling (1998) "Deardorff's Condition for Factor Price Equalization in the Two-factor Case" (mimeo, Kyoto University). Both authors proved the same theorem. However, my work is entirely independent, and through private conversation with Ufuk Demiroglu, I learned that both proofs were long and not easily understandable. I also learned that Demiroglu's mimeo is unfinished.

[5] Arguably, the evidence is not overwhelming. For instance, other factors, such as land, are left out of the test. However, I believe that their work is the first step in the right direction.

performance of HOV among developed countries[6] is not due to the failure of FPE that is caused by very different factor endowments.

2. The Setup, and the Lemma

Consider a multi-dimensional Heckscher-Ohlin setup. Suppose there are f factors, n sectors, and m countries. Index the factors, sectors and countries by j, i and c respectively. Each sector uses some or all of the factors as inputs, and produces one distinct consumption good as the output. For each sector i, the production technology is identical across countries, and characterized by constant returns to scale. All the domestic and international markets are perfectly competitive. Factors are mobile within a country, but immobile internationally.

If, in the above model, factors are allowed instead to move freely across country borders, the economy becomes the "integrated world economy" (IWE). As argued by Dixit and Norman (1980), FPE can be achieved without international factor mobility if and only if the world outputs of the IWE can be replicated using the IWE production techniques while fully employing each factor in each country. Call this condition "the FPE condition".

Let v be an $n \times f$ matrix that has as its rows the factor usage vectors of the sectors of the IWE; i.e. v_{ij} is the amount of factor j demanded by sector i in the IWE. Call v the "IWE factor usage matrix." Let V be an $m \times f$ matrix that has as its rows the factor endowment vectors of all the countries; i.e. V_{cj} is country c's endowment of factor j. Call V the "factor endowment matrix." Let u_n be a $1 \times n$ row vector of ones, and u_m a $1 \times m$ row vector of ones. Since all factors are fully employed in the IWE, the individual sectors' demands for each factor must add up to the world endowment of that factor. In mathematical notation, $u_n\, v = u_m\, V$ (v is said to be an "equal-sum matrix" to V).

[6] Davis and Weinstein (1998) find good empirical support for HOV among developed countries after considering such extensions as technological differences and non-FPE.

One way of stating the FPE condition formally is to define the operator “-FPE-” as follows:

Definition (-FPE-): v-FPE-V if and only if there exists an $m \times n$ matrix, λ, (referred to as the “FPE matrix”) such that:

$$(1) \quad \lambda \, v = V$$

$$(2) \quad \forall \; c=1,2\ldots m \text{ and } i = 1,2\ldots n, \;\; \lambda_{ci} \in [0, 1]$$

$$(3) \quad u_m \, \lambda = u_n$$

The FPE matrix λ tells us how, to achieve FPE, the production of the world outputs of the IWE should be distributed across the *m* countries. Namely, λ_{ci} is the share of the IWE output of good *i* that country *c* has to produce using the IWE production technique. Condition (1) then says that after such a distribution, each factor in each country has to be fully employed. Condition (2) says that all the shares have to be between 0 and 1. Condition (3) says that for a given good *i*, the shares of all the countries have to add up to 1.

It turns out that the operator “-FPE-“ has the following important property:

Lemma: Let the $n \times f$ matrix v, $q \times f$ matrix M, and $m \times f$ matrix V be equal sum matrices such that v-FPE-M and M-FPE-V. Then v-FPE-V; i.e. the operator “-FPE-“ is transitive.

Proof: By definition of –FPE-, there exist a $q \times n$ FPE matrix λ_1 and an $m \times q$ FPE matrix λ_2 such that

$$\lambda_1 \, v = M$$

$$\lambda_2 \, M = V$$

$$u_q \, \lambda_1 = u_n, \qquad u_m \, \lambda_2 = u_q \tag{4}$$

The first 2 equations yield:

$$\lambda_2 \, \lambda_1 \, v = V \tag{5}$$

Let $\lambda_3 = \lambda_2 \, \lambda_1$. It can be shown that

$$\lambda_3 \text{ is an FPE matrix} \tag{6}$$

(i)By definition of λ_3

$$\lambda_{3ci} = \sum_p \lambda_{2cp}\ \lambda_{1pi} \quad \forall\ c=1,2,\ldots m,\ i=1,2,\ldots n.$$

For each *i*, the sum of λ_{1pi} over *p* is 1 since λ_1 is an FPE matrix. Therefore λ_{3ci} is the weighted average of λ_{2cp}, all of which lie between 0 and 1. Thus each element of the matrix λ_3 lies between 0 and 1.

(ii)By (4), $u_m\ \lambda_3 = u_m\ \lambda_2\ \lambda_1 = u_q\ \lambda_1 = u_n$

Together with (5), (i) and (ii) establish claim (6). Because *v* and *V* are equal sum matrices, *v*-FPE-*V*. QED

Next, I need to define "lens", and formalize the "lens condition". The definitions below follow Deardorff (1994).

Definition (lens): Let *a* be an $n \times f$ matrix. The lens spanned by the row vectors of *a* is defined to be:

$$L(a) = \{\ x \in R^f : x = b\ a \text{ for some } 1 \times n \text{ vector } b \text{ such that } b_k \in [0,1]\ \forall\ k = 1,2,\ldots,n\}$$

When there are only 2 factors, the row vectors of *a* are 2-dimenstional, and *L(a)* can be represented by a polygon in the factor endowment box. Each side of the polygon corresponds to one row vector of *a*, and the polygon is symmetric about the diagonal of the box. [7]

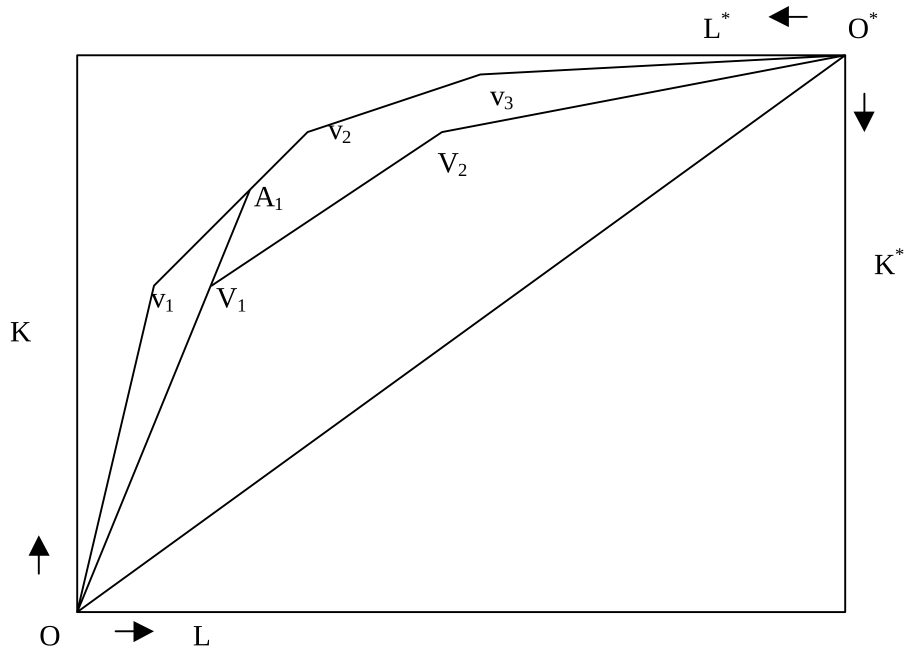
Definition (lens condition): Let *v* be an $n \times f$ IWE factor usage matrix and *V* an $m \times f$ factor endowment matrix; let *v* and *V* be equal sum matrices. Matrices *v* and *V* satisfy the lens condition if and only if $L(V) \subseteq L(v)$.

3. The Theorem, and the Intuition

[7] See, for example, Figure 2 of Deardorff (1994).

Theorem: Let v be an $n \times 2$ IWE factor usage matrix and V an $m \times 2$ factor endowment matrix; let v and V be equal sum matrices. If $L(V) \subseteq L(v)$, then v-FPE-V.

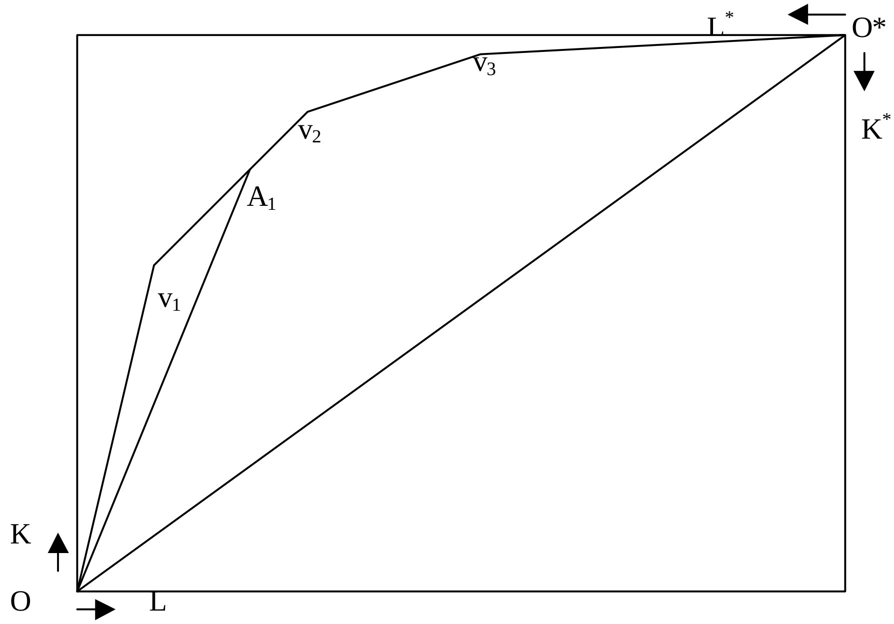
Before presenting the formal proof, I consider a case with 3 countries and 4 sectors (i.e. $m = 3$ and $n = 4$). The example not only illustrates the gist of the proof, but also reveals the economic intuition behind it.



Graph 1a

Call the 2 factors capital and labor, and suppose the world endowment of capital and labor are V_{wk} and V_{wl} respectively. A 2×2 box can be drawn with width V_{wk} and length V_{wl}, as in Graph 1a. In the graph, Ov_1, v_1v_2, v_2v_3 and v_3O^* are the row vectors of the 4×2 IWE factor usage matrix v, and represent good 1 through good 4 respectively. The polygon $Ov_1v_2v_3O^*$ is the upper half of the lens $L(v)$.[8] Similarly, OV_1, V_1V_2, and V_2O^* are the row vectors of the 3×2 factor endowment matrix V, and represent country 1 through country 3 respectively. The polygon $OV_1V_2O^*$ is the upper half of the lens $L(V)$.

[8] Because the lens $L(v)$ is symmetric about the diagonal of the box, I focus on what happens above the diagonal.



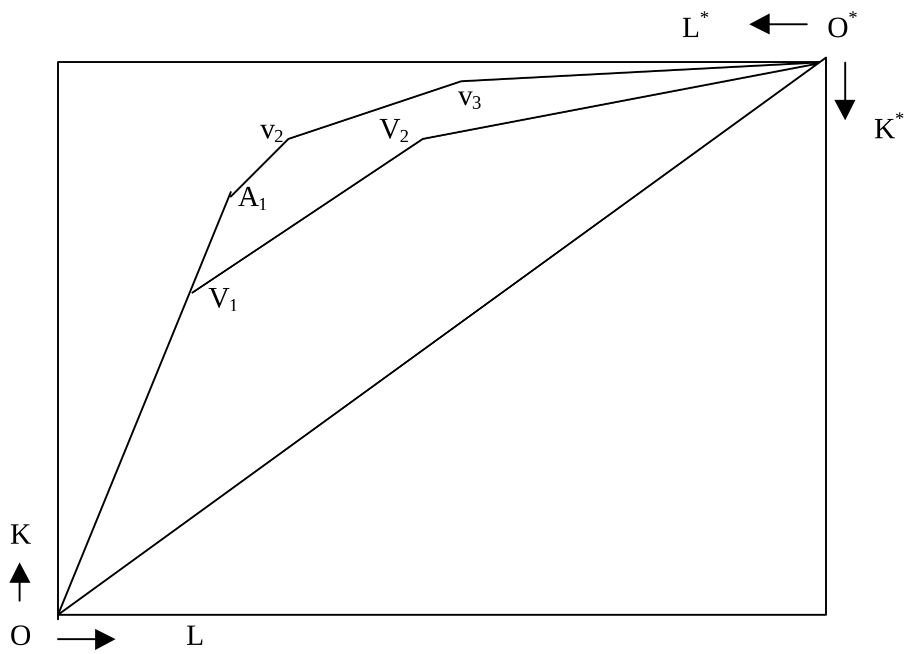
Graph 1b

The logic of the proof is as follows. First, I construct an intermediate lens *L(M)* such that *L(V)* Í *L(M)* Í *L(v)*. Second, I show that *v*-FPE-*M*. Third, I use a proof-by-induction argument to show that *M*-FPE-*V*. Finally, I apply the Lemma to show that *v*-FPE-*V*.

Extend OV_1 so that it intersects the polygon $Ov_1v_2v_3O^*$ at A_1. The polygon $OA_1v_2v_3O^*$ is the upper half of a new lens spanned by OA_1, A_1v_2, v_2v_3 and v_3O^*. Denote this lens by *L(M)*. Clearly *L(M)* lies within *L(v)* but contains *L(V)*. Also, make the standard proof-by-induction assumption that the Theorem holds for the case of $m = 2$. Call this Assumption (X).

First of all, imagine a new world with the same sectors but 4 fictional countries, as shown in Graph 1b. In this world, the IWE factor usage matrix is still *v*, but the factor endowment matrix is *M*. Namely, the factor endowments of countries 1', 2', 3' and 4' are OA_1, A_1v_2, v_2v_3 and v_3O^* respectively. It is straightforward to achieve FPE in this world. Country 1' produces good 1 and a share (v_1A_1/v_1v_2) of the IWE output of good 2. Country 2' produces the remaining share

(A_1v_2/v_1v_2) of the IWE output of good 2. Country 3' produces good 3 and country 4' produces good 4.[9] Therefore *v*-FPE-*M*.



Graph 1c

Second, consider another world shown in Graph 1c, where there are 4 fictional sectors and the same 3 original countries. The factor endowment matrix here is still *V*, but the IWE factor usage matrix is *M*. Namely, the factor usage vectors of sectors 1', 2', 3' and 4' are OA_1, A_1v_2, v_2v_3 and v_3O^* respectively.[10] To achieve FPE in this world, I could assign country 1, the most capital-abundant country, to produce nothing but good 1', the most capital-intensive good. By the construction of this new world, the factor usage vector of good 1' is proportional to, and at least as large as, the factor endowment vector of country 1, and therefore the factors of country 1 are fully employed. Now all I need to do is to assign the production of the remaining IWE outputs (whose factor usage vectors are V_1A_1, A_1v_2, v_2v_3 and v_3O^*) to the remaining 2 countries (V_1V_2 and V_2O^*), and I know I can do that by Assumption (X). Therefore *M*-FPE-*V*.

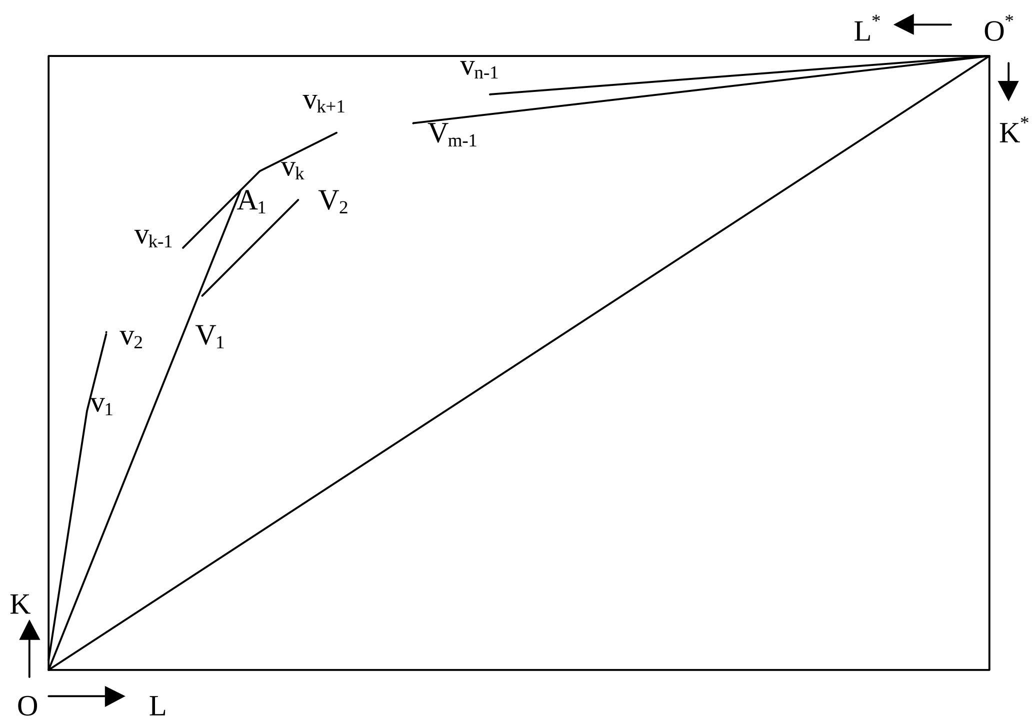
[9] The corresponding FPE matrix is in Appendix A. Of course, this is only one of the many possible allocations.

[10] Notice that the row vectors of *M* play the role of fictional countries in proving *v*-FPE-*M*, and now they play the role of fictional sectors.

Finally, with *v*-FPE-*M* and *M*-FPE-*V*, the Lemma can be applied to show that *v*-FPE-*V*.

4. The Proof

The proof is closely related to Graph 2. Denote the row vectors of the $n \times 2$ IWE factor usage matrix *v* by Ov_1, v_1v_2, …, $v_{n-1}O^*$, as in Graph 2, and similarly those of the $m \times 2$ factor endowment matrix *V* by OV_1, V_1V_2, …, $V_{m-1}O^*$.



Graph 2

The proof is by induction.

(A). The Theorem holds for all *n* and *m* =1. When there is only one country, the country is the world itself, therefore the Theorem holds trivially.

(B). Assume that the Theorem holds for all *n* and *m*-1. If it can be shown that the Theorem holds for all *n* and *m*, I am home.

As in Graph 2, extend OV_1 to intersect the polygon $Ov_1v_2 \ldots v_{n-1}O^*$ at A_1. Without loss of generality, assume that point A_1 is on $v_{k-1}v_k$. Let $\alpha = v_{k-1}A_1/v_{k-1}v_k$ ($\alpha \in [0, 1)$). It is straightforward to check that the $(n\text{-}k+2) \times 2$ matrix with row vectors OA_1, A_1v_k, v_kv_{k+1}, … $v_{n-1}O^*$ (denote the matrix by M) is an "equal-sum" matrix to both v and V.

(C) *v*-FPE-*M*.

Proof: By examining Graph 2, the following $(n\text{-}k+2) \times$ n matrix can be constructed:

$$l = \begin{pmatrix} 1 & .. & 1 & a & 0 & .. & 0 & 0 \\ 0 & .. & 0 & 1-a & 0 & .. & 0 & 0 \\ 0 & .. & 0 & 0 & 1 & .. & 0 & 0 \\ .. & .. & .. & .. & 0 & 1 & .. & 0 \\ 0 & .. & 0 & 0 & 0 & 0 & .. & 1 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\}2 \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\}n-k \end{matrix}$$

$$\underbrace{\qquad\qquad}_{k-1} \qquad\qquad \underbrace{\qquad\qquad\qquad}_{n-k}$$

Noting that $A_1v_k = (1 - a\ ) v_{k\text{-}1}v_k$, it is easy to check that λ is the FPE matrix needed so that *v*-FPE-*M*; namely, $l\ v = M$, $u_{n\text{-}k+2}\ \ l = u_n$, and l_{ci} Î $[0,1]$ " c, i. QED

(**D) *M*-FPE-*V***

Proof: Let $b = OV_1/OA_1$, b Î $(0,1]$. Then $V_1A_1 = (1 - b\ ) OA_1$.

Consider the following $(n\text{-}k+2) \times 2$ matrix M_1 and $(m\text{-}1) \times 2$ matrix W:

$$M_1 = \begin{pmatrix} V_1A_1 \\ A_1v_k \\ .. \\ v_{n-1}O^* \end{pmatrix}, \text{ and } W = \begin{pmatrix} V_1V_2 \\ V_2V_3 \\ .. \\ V_{m-1}O^* \end{pmatrix}.$$

It is easy to check that they are "equal-sum" matrices. Therefore, by (B), there exists an $(m\text{-}1) \times n$ FPE matrix λ' such that M_1 -FPE-W.

Partition λ' as :

$$l' = \big(\underbrace{l_1}_{1} \qquad \underbrace{l_{-1}}_{n-1}\big)$$

Since λ' is an FPE matrix, we have:

$$u_{m\text{-}1}\ \ l_1 = 1$$

$$u_{m-1} \; l_{-1} = u_{n-1} \qquad (7)$$

Now I want to show that the following *m* ´ *n* matrix is the FPE matrix needed so that *M*-FPE-*V*:

$$l'' = \begin{pmatrix} \underbrace{\begin{matrix} b \\ (1-b)\, l_1 \end{matrix}}_{1} & \underbrace{\begin{matrix} 0 \\ l_{-1} \end{matrix}}_{n-1} \end{pmatrix}$$

(I) $$l''\, M = \begin{pmatrix} \underbrace{\begin{matrix} b \\ (1-b)\, l_1 \end{matrix}}_{1} & \underbrace{\begin{matrix} 0 \\ l_{-1} \end{matrix}}_{n-1} \end{pmatrix} M = \begin{pmatrix} bOA_1 \\ l' M_1 \end{pmatrix} = \begin{pmatrix} OV_1 \\ W \end{pmatrix} = V,$$

where the second equality uses the fact that *(1 – b)* $OA_1 = V_1A_1$, and the third equality follows from the definition of β and that λ' is an FPE matrix.

(II) $$u_m\, l'' = (1 \quad u_{m-1}) \begin{pmatrix} \underbrace{\begin{matrix} b \\ (1-b)\, l_1 \end{matrix}}_{1} & \underbrace{\begin{matrix} 0 \\ l_{-1} \end{matrix}}_{n-1} \end{pmatrix}$$

$$= \left(b + (1-b)(u_{m-1}\, l_1) \quad u_{m-1}\, l_{-1}\right) = \left(1 \quad u_{n-1}\right) = u_n$$

where the third equality uses equation (7).

(III) It is straightforward to check that each element of λ'' is in [0,1].

By (I) (II) and (III), I have shown that λ'' is the FPE matrix needed so that *M*-FPE-*V*. QED

(E) v*-FPE-*V

Proof: The Lemma, together with (C) and (D), implies that *v*-FPE-*V*. QED

Conclusion

This paper proves the sufficiency of the lens condition for FPE in the case of 2 factors. Together with the work of Deardorff (1994) and Demiroglu and Yun (1997), this result helps form a clear picture: the lens condition is sufficient for factor price equalization in the case of 2 countries,[11] 2 factors, or 2 goods. [12]When there are more than 2 factors, Demiroglu and Yun (1997) show that the lens condition may be insufficient for FPE by providing 2 counter examples, both involving 3 factors.

Moreover, this result justifies the use of the 2-dimensional factor endowment box to analyze factor price equalization for an arbitrary number of countries and goods. In addition, this result serves as the theoretical foundation for meaningful empirical work. One such example is Debaere and Demiroglu (1998), who assess the similarity of country endowment proportions using 2 factors, capital and labor.

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Appendix A

$$l = \begin{pmatrix} 1 & \frac{v_1 A_1}{v_1 v_2} & 0 & 0 \\ 0 & \frac{A_1 v_2}{v_1 v_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$